

On the inverse scattering method(2)

その他（別言語等） のタイトル	逆錯乱法について(2)
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On the Inverse Scattering Method II

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Abstract

The reconstruction of the confining potentials from the bound state data has been performed for $N=6, 8$ and 10 . The reduced form of the determinant is derived to overcome some computational difficulties and to improve the inverse scattering method in one dimension for large N .

I Introduction

In the previous paper¹⁾ we reconstructed the quarkonium potential by the inverse scattering method for $N=4$, following Thacker, Quigg and Rosner (QRT)²⁾. We want to get an interquark potential including new experimental data, but in the practical application we have to calculate the determinants for large N , which are rather difficult because of large number of elements.

In this paper we derive the reduced form of determinant of the matrix and apply it to the well known cases, i.e. harmonic oscillator and square well potentials in one dimension. The mathematical aspect of the spectral theory and derivation of the reduced form of the determinant are given in appendices.

II Mathematical formalism of inverse scattering method

In this section we first summarize the inverse scattering method for the Schrödinger equation in one dimension. It is well known that the Schrödinger equation appears as a characteristic eigenvalue problem for solving nonlinear Korteweg-de Vries equation, defining the scattering data for the given initial wave form $U(x; t=0)$. In the theory of nonlinear waves, the inverse scattering problem is solved in order to find the wave amplitude $U(x; t)$ at a later time $t > 0$, based on the scattering data for $U(x; t=0)$, i.e. the initial conditions.

The solution of the inverse scattering problem is obtained by solving an integral equation, called Gelfand-Levitan-Marchenko (GLM) equation, for a function $K(x, y)$. If the expected potential is assumed to be reflectionless, this has a simple form³⁾

$$K(x, y) + \sum_{m=1}^N C_m^2 e^{-\kappa_m(x+y)} + \sum_{m=1}^N C_m^2 \int_x^\infty K(x, z) e^{-\kappa_m(y+z)} dz = 0 \quad (2-1)$$

where $-\kappa_m^2$ are the bound state energy and C_m are the asymptotic coefficients of the normalized wave functions⁴⁾.

In order to solve Eq. (2-1), it is convenient to put $K(x, y)$ in the form

$$K(x, y) = - \sum_{m=1}^N C_m e^{-\kappa_m y} \Psi_m(x) \quad (2-2)$$

with unknown functions Ψ_m . Then Eq. (2-1) becomes a set of equations for Ψ_m

$$\Psi_m(x) + \sum_{n=1}^N C_m C_n \frac{e^{-(\kappa_m + \kappa_n)x}}{\kappa_m + \kappa_n} \Psi_n(x) - C_m e^{-\kappa_m x} = 0 \quad (2-3)$$

This is rewritten as

$$\sum_{n=1}^N B_{mn} \Psi_n(x) = \lambda_m(x) \quad (2-4)$$

where

$$B_{mn} = \delta_{mn} + \frac{\lambda_m \lambda_n}{\kappa_m + \kappa_n} \quad (2-5)$$

and

$$\lambda_m = C_m e^{-\kappa_m x} \quad (2-6)$$

Noticing the identity $\frac{\partial}{\partial x} \frac{\lambda_m \lambda_n}{\kappa_m + \kappa_n} = -\lambda_m \lambda_n$,²⁾

we can express the solution of Eq. (2-4)

$$\Psi_n(x) = -\frac{1}{\lambda_n} \frac{\det B^{(n)}}{\det B} \quad (2-7)$$

where we denote by $B^{(n)}$ the matrix, obtained from B by replacing the n -th column with its derivative. After determining the function $K(x, y)$ we finally obtain the potential $U(x)$ from the relation.

$$\begin{aligned} U(x) &= -2 \frac{d}{dx} K(x, x) \\ &= -2 \frac{d}{dx} \frac{\frac{d}{dx}(\det B)}{\det B} \\ &= -2 \frac{d^2}{dx^2} \ln(\det B) \end{aligned} \quad (2-8)$$

with the reduced form

$$\det B = \sum_{T \in S} \left[\exp \left(- \sum_{m \in T} 2\kappa_m x \right) \prod_{n \in N-T} \left| \frac{\kappa_m + \kappa_n}{\kappa_m - \kappa_n} \right| \right] \quad (2-9)$$

where S is a set of all subsets T of N -numbers and $N-T$ is the complement of T .

III Construction of some confining potentials in one dimension

In this section we apply the general scheme of the inverse scattering method obtained in last section to derive the potential from the knowledge of a set of bound state energies of the form $\kappa_n^2 = E_0 - E_n$ ($n=1, 2, \dots, N$)²⁾. Here E_n is the energy eigenvalue of the n -th bound state for the symmetric potential $u(x) = u(-x)$ and E_0 is a fitting parameter. Then the corresponding reflectionless potential should be given by

$$u(x) = E_0 - 2 \frac{d^2}{dx^2} \ln(\det B) \quad (3-1)$$

As a trial, we choose E_0 of the form

$$E_0 = (E_N + E_{N+1})/2 + \epsilon \Delta E \quad (3-2)$$

with

$$\epsilon = -1, 0, 1 \quad \text{and} \quad \Delta E = (E_{N+1} - E_N)/4.$$

When E_n 's are taken as the eigenvalues of the Hamiltonian of the harmonic oscillator, the resulting potentials are plotted in Fig. 1. The results for E_n of square well case are shown in Fig. 2.

For the harmonic oscillator potential $u(x) = m\omega^2 x^2/2$, we take $(\hbar/m\omega)^{1/2}$ as a unit of the X -axis and $\hbar\omega/2$ as a unit of the Y -axis. In order to get a good approximation in the range $-4.0 < X < 4.0$ we must take more than 10 bound states as E_n . For $N < 10$ we can not reconstruct the potential sufficiently well at $X=4.0$.

For the square well potential with $U(x)=0$ for $-a < x < a$ and with $U(x)=\infty$ elsewhere, we take Y as the potential height in the unit of $\hbar^2/(2ma^2)$ and X as the length x in the unit of $a/3.5$.

We note that we have the better result for the reconstructed potential for larger N .

VI Discussions and conclusions

In the previous section we have seen the validity of using the reduced form for the determinant B with large dimension N . We have applied it to the recon-

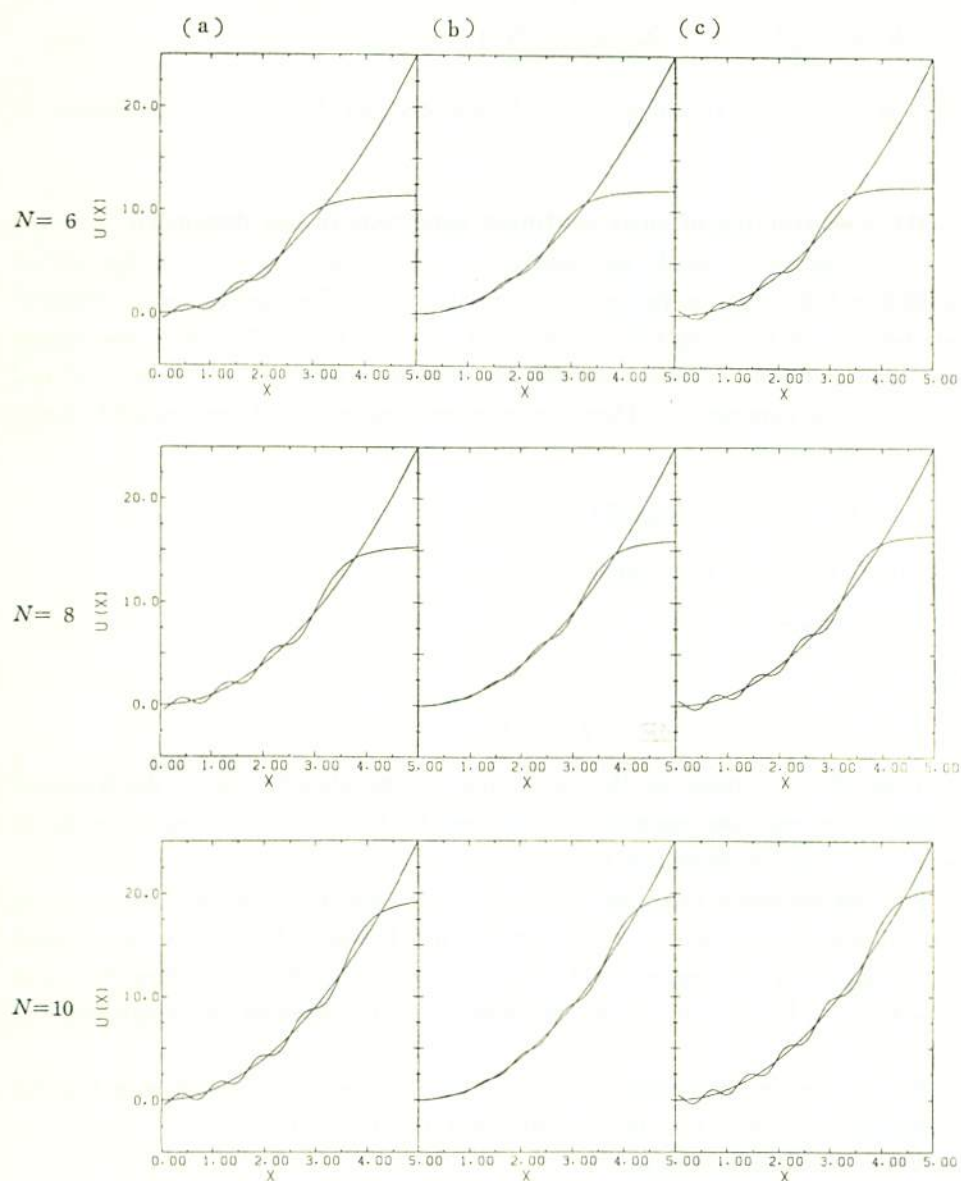


Fig. 1 Reconstructed harmonic oscillator potentials in one dimension.

The first row: $N=6$; the second row: $N=8$;

the third row: $N=10$,

(a) column: $\epsilon=-1$ in Eq. (3-2); (b) column: $\epsilon=0$;

(c) column: $\epsilon=1$.

The exact potential is shown for comparison.

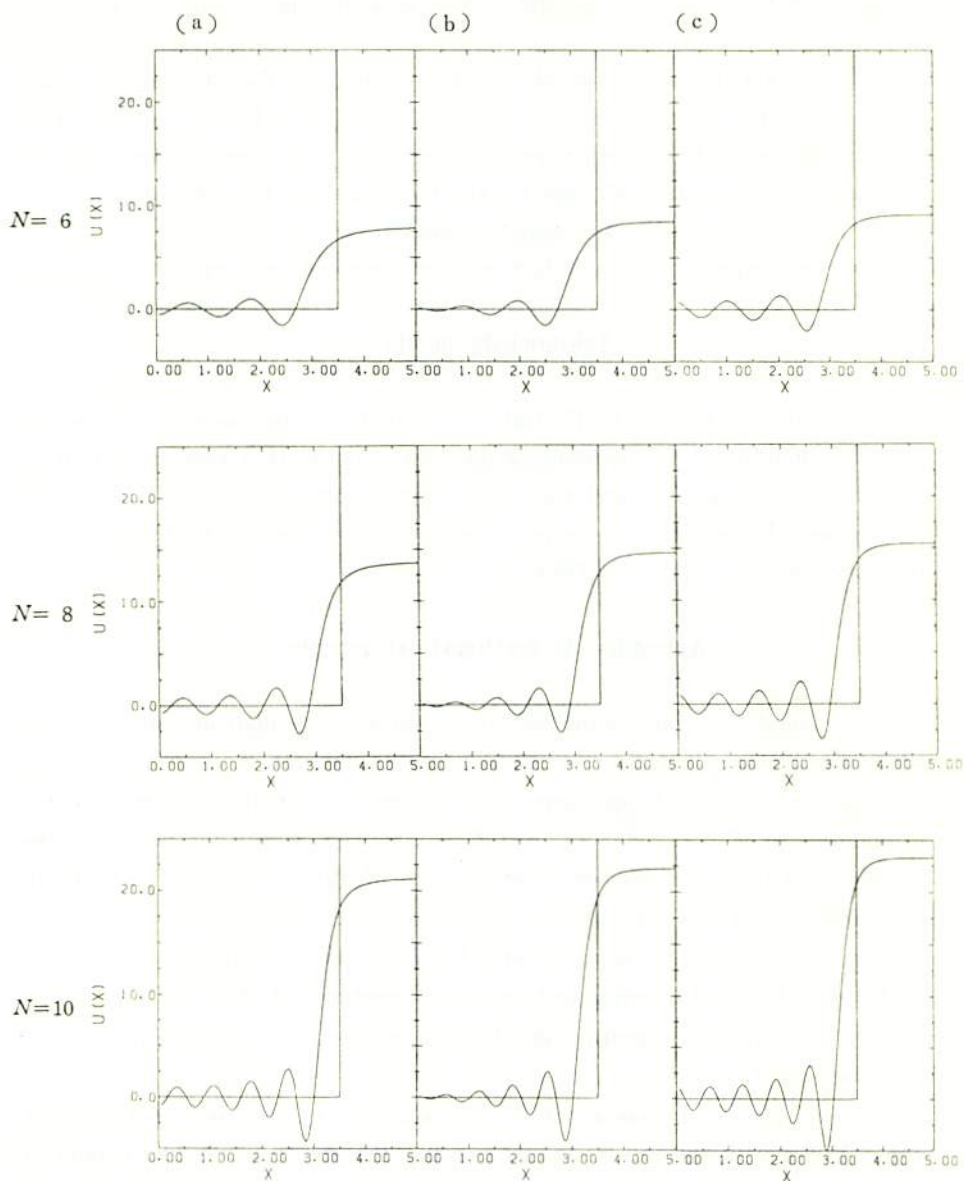


Fig. 2 Reconstructed square well potentials in one dimension.
See caption to Fig. 1 for details.

struction of known potentials from the knowledge of their discrete energy eigenvalues, making use of the general scheme of inverse scattering method for reflectionless potentials.

It is shown that the reconstructed potentials reproduce the original form well when the fitting parameter E_0 is chosen to be $E_0 = (E_N + E_{N+1})/2$. For $N=5$ Asthana and Kamal⁵⁾ tried to reconstruct various potentials in one dimension. We find that the method works well even for the potentials not reflectionless when we have large N and consider the appropriate region of x .

Further application to our formalism will be discussed in a separate paper.

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Appendix A: mathematical aspects

In this appendix we are going to discuss some mathematical aspects of the method in the spectral theory.

We denote a complex Hilbert space by X , a set of selfadjoint closed operator from X to itself by $\Gamma(X)$. For an A in $\Gamma(X)$, we can classify the set of complex numbers $\{z \in C\}$ into three classes, where I is an identity operator, C and R are the set of complex and real numbers⁶⁾.

i) Resolvent set $\rho(A)$: an operator $(A - zI)$ is bijective. In this case $R_z(A) = (A - zI)^{-1}$ is a bounded operator and called the resolvent of A .

ii) Set of continuous spectrum $\sigma_c(A)$: an operator $(A - zI)$ is injective but not surjective. $\sigma_c(A) \subset R$.

iii) Set of point spectrum $\sigma_p(A)$: an operator $(A - zI)^{-1}$ does not exist. $\sigma_p(A) \subset R$. The sum of the sets σ_c and σ_p is called the spectrum of the operator A . Then the resolvent set is the complement of the spectrum.

On the other hand, the theory of spectral resolution tells us that a representation of the form⁷⁾

$$f(A) = \int_{-\infty}^{\infty} f(\lambda) dE(\lambda) \quad (A-1)$$

is possible for an arbitrary function f and $E(\lambda)$ ($\lambda \in R$) is a spectral function of A .

Applying this theorem to the function $f_u(\lambda) = (\lambda - \mu)^{-1}$, we get the following representation of the resolvent in terms of the spectrum of A .

$$\begin{aligned} R_\mu(A) &= (A - \mu I)^{-1} \quad \mu \in \rho(A) \\ &= \int (\lambda - \mu)^{-1} dE(\lambda) \\ &= -i \int_0^{+\infty} dt \int e^{i(\lambda - \mu)t} dE(\lambda). \end{aligned} \quad (\text{A-2})$$

Now we consider an inhomogeneous differential equation

$$(L - \lambda)\Psi(x) = [-\partial_x^2 + U(x) - \lambda]\Psi = h(x) \quad (\text{A-3})$$

involving the Schrödinger operator L in one dimension. The solution of Eq. (A-3) is obtained in the form.

$$\Psi(x) = (L - \lambda)^{-1} h(x) = (R_\lambda h)(x) = \int_{-\infty}^{\infty} dy R_\lambda(x, y) h(y), \quad (\text{A-4})$$

where $R_\lambda(x, y)$ is the integral kernel representation of the resolvent $R_\lambda = (L - \lambda I)^{-1}$ ($\lambda \in \rho(L)$)³⁾.

Comparing Eq. (A-2) and Eq. (A-4), we see that the differential equation (A-3) is solved immediately, if we get the inverse operator R_λ , or equivalently, the spectra of the linear operator L forming up the resolvent kernel.

The ordinary scattering problem is to derive the expressions of the scattering data by which we mean the reflection and transmission coefficients, for a given scattering potential $u(x)$. In contrast, the inverse scattering method enables us to derive the scattering potential $u(x)$ from the given knowledge of the scattering data, which include the point spectra by the inverse spectral theory as the bound state poles.

Appendix B: derivation of Eq. (2-9)

We decompose the matrix B of Eq. (2-5) into δ_{mn} and $\lambda_m \lambda_n / (\kappa_m + \kappa_n)$ at the m -th row, resulting in the sum of 2^N elements.

$$\det B = \sum_{T \in S} \det \left(\frac{\lambda_m \lambda_n}{\kappa_m + \kappa_n} \right)_{m, n \in T} \quad (\text{B-1})$$

$$= \sum_{T \in S} \left[\prod_{m \in T} \lambda_m^2 \det \left(\frac{1}{\kappa_m + \kappa_n} \right)_{m, n \in T} \right] \quad (\text{B-2})$$

where S is a set of all subsets T of N -numbers $\{1, 2, \dots, N\}$, including the null set and the full set. Making use of the formula³⁾,

$$\det \left(\frac{1}{\kappa_i + \kappa_j} \right)_{i, j \in K} = \prod_{i < j}^K (\kappa_i - \kappa_j)^2 / \prod_{i, j}^K (\kappa_i + \kappa_j) \quad (\text{B-3})$$

and

$$C_n^2 = 2\kappa_n \prod_{m \neq n}^N \left| \frac{\kappa_n + \kappa_m}{\kappa_n - \kappa_m} \right| \quad \text{at } t = 0, \quad (\text{B-4})$$

we can express $\det B$ as follows,

$$\det B = \sum_{T \in S} \left(\prod_{m \in T} \lambda_m^2 \prod_{\substack{m < n \\ m, n \in T}} (\kappa_m - \kappa_n)^2 / \prod_{m, n \in T} (\kappa_m + \kappa_n) \right) \quad (\text{B-5})$$

$$= \sum_{T \in S} \left[e^{-\left(\sum_{m \in T} 2\kappa_m\right)x} \prod_{m \in T} \left((\kappa_m + \kappa_m) \prod_{\substack{m \neq n \\ n=1}}^N \left| \frac{\kappa_m + \kappa_n}{\kappa_n - \kappa_m} \right| \right) \right. \\ \left. \times \prod_{\substack{m < n \\ m, n \in T}} (\kappa_m - \kappa_n)^2 / \prod_{m, n \in T} (\kappa_m + \kappa_n) \right] \quad (\text{B-6})$$

$$= \sum_{T \in S} \left(e^{-\left(\sum_{m \in T} 2\kappa_m\right)x} \prod_{\substack{m \in T \\ n \in N-T}} \left| \frac{\kappa_m + \kappa_n}{\kappa_m - \kappa_n} \right| \right) \quad (\text{B-7})$$

where $N-T$ is the complement of the subset T ⁹⁾.

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